



Curriculum Framework

# AP<sup>®</sup> Calculus AB and AP<sup>®</sup> Calculus BC

2016–2017



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The College Board strongly encourages educators to make equitable access a guiding principle for their AP programs by giving all willing and academically prepared students the opportunity to participate in AP. We encourage the elimination of barriers that restrict access to AP for students from ethnic, racial, and socioeconomic groups that have been traditionally underserved. Schools should make every effort to ensure their AP classes reflect the diversity of their student population. The College Board also believes that all students should have access to academically challenging course work before they enroll in AP classes, which can prepare them for AP success. It is only through a commitment to equitable preparation and access that true equity and excellence can be achieved.

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The *AP Calculus AB and AP Calculus BC Curriculum Framework* is designed to provide educators with a first look at essential information needed to understand the design and intent of the revised AP Calculus courses in advance of their implementation in schools in the 2016-17 academic year. Please be advised that the information contained in this document is subject to change. Final course and exam information will be available in the *AP Calculus AB and AP Calculus BC Course and Exam Description*, which will be published in early 2016.

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# Introduction

## About AP Calculus AB and AP Calculus BC

Building enduring mathematical understanding requires students to understand the *why* and *how* of mathematics in addition to mastering the necessary procedures and skills. To foster this deeper level of learning, AP<sup>®</sup> Calculus is designed to develop mathematical knowledge conceptually, guiding students to connect topics and representations throughout each course and to apply strategies and techniques to accurately solve diverse types of problems.

AP Calculus includes two courses, AP Calculus AB and AP Calculus BC, which were developed in collaboration with college faculty. The curriculum for AP Calculus AB is equivalent to that of a first-semester college calculus course, while AP Calculus BC is equivalent to a first-semester college calculus course and the subsequent single-variable calculus course. Calculus BC is an extension of Calculus AB rather than an enhancement; common topics require a similar depth of understanding. Both courses are intended to be challenging and demanding, and each is designed to be taught over a full academic year. The *AP Calculus AB and AP Calculus BC Curriculum Framework* specifies the curriculum — what students must know, be able to do, and understand — for both courses.

AP Calculus AB is structured around three big ideas: limits, derivatives, and integrals and the Fundamental Theorem of Calculus. AP Calculus BC explores these ideas in additional contexts and also adds the big idea of series. In both courses, the concept of limits is foundational; the understanding of this fundamental tool leads to the development of more advanced tools and concepts that prepare students to grasp the Fundamental Theorem of Calculus, a central idea of AP Calculus.

The Mathematical Practices for AP Calculus (MPACs), presented in this curriculum framework, explicitly describe the practices students will apply to accomplish the learning objectives of the courses and build conceptual understanding. As students explore the subject matter of AP Calculus, they learn to cultivate and apply the MPACs, thus developing the problem-solving skills required to engage deeply with mathematical concepts.

## Changes to the AP Calculus Courses and Exams

This curriculum framework presents changes to the AP Calculus courses and exams for implementation in the 2016-17 academic year. Changes to the courses include the following:

- ▶ In order to promote conceptual understanding, an Understanding by Design (Wiggins and McTighe) structure replaces the topic list.
- ▶ Course content is directly and deliberately aligned with demonstrable learning objectives.
- ▶ Essential mathematical practices are made explicit.
- ▶ L'Hospital's Rule has been added to Calculus AB.

## Introduction

- ▶ The limit comparison test, absolute and conditional convergence, and the alternating series error bound have been added to Calculus BC.
- ▶ No topics have been deleted from the existing AP Calculus program.

The AP Calculus AB and BC Exams continue to share the same format, which consists of a multiple-choice section and a free-response section. The structure of the free-response section has not changed. Changes to the multiple-choice section are as follows:

- ▶ Multiple-choice questions now have four answer choices instead of five.
- ▶ The distribution of questions and relative timing have been adjusted based on feedback from teachers and administrators, although the number of questions and the total allotted time remain the same (45 questions in 105 minutes).
  - › Part A of the multiple-choice section, during which calculator use is not permitted, consists of 30 questions in 60 minutes.
  - › Part B of the multiple-choice section, during which calculator use is required, consists of 15 questions in 45 minutes.



# The Curriculum Framework

## Overview

Based on the Understanding by Design (Wiggins and McTighe) model, this curriculum framework is intended to provide a clear and detailed description of the course requirements necessary for student success. It presents the development and organization of learning outcomes from general to specific, with focused statements about the content knowledge and understandings students will acquire throughout the course.

The **Mathematical Practices for AP Calculus (MPACs)**, which explicitly articulate the behaviors in which students need to engage in order to achieve conceptual understanding in the AP Calculus courses, are at the core of this curriculum framework. Each concept and topic addressed in the courses can be linked to one or more of the MPACs.

This framework also contains a **concept outline**, which presents the subject matter of the courses in a table format. Subject matter that is included only in the BC course is indicated with blue shading. The components of the concept outline are as follows:

- ▶ **Big ideas:** The courses are organized around big ideas, which correspond to foundational concepts of calculus: limits, derivatives, integrals and the Fundamental Theorem of Calculus, and (for AP Calculus BC) series.
- ▶ **Enduring understandings:** Within each big idea are enduring understandings. These are the long-term takeaways related to the big ideas that a student should have after exploring the content and skills. These understandings are expressed as generalizations that specify what a student will come to understand about the key concepts in each course. Enduring understandings are labeled to correspond with the appropriate big idea.
- ▶ **Learning objectives:** Linked to each enduring understanding are the corresponding learning objectives. The learning objectives convey what a student needs to be able to do in order to develop the enduring understandings. The learning objectives serve as targets of assessment for each course. Learning objectives are labeled to correspond with the appropriate big idea and enduring understanding.
- ▶ **Essential knowledge:** Essential knowledge statements describe the facts and basic concepts that a student should know and be able to recall in order to demonstrate mastery of each learning objective. Essential knowledge statements are labeled to correspond with the appropriate big idea, enduring understanding, and learning objective.

Further clarification regarding the content covered in AP Calculus is provided by **examples** and **exclusion statements**. Examples are provided to address potential inconsistencies among definitions given by various sources. Exclusion statements identify topics that may be covered in a first-year college calculus course but are not assessed on the AP Calculus AB or BC Exam. Although these topics are not assessed, the AP Calculus courses are designed to support teachers who wish to introduce these topics to students.

## Mathematical Practices for AP Calculus (MPACs)

The Mathematical Practices for AP Calculus (MPACs) capture important aspects of the work that mathematicians engage in, at the level of competence expected of AP Calculus students. They are drawn from the rich work in the National Council of Teachers of Mathematics (NCTM) Process Standards and the Association of American Colleges and Universities (AAC&U) Quantitative Literacy VALUE Rubric. Embedding these practices in the study of calculus enables students to establish mathematical lines of reasoning and use them to apply mathematical concepts and tools to solve problems. The Mathematical Practices for AP Calculus are not intended to be viewed as discrete items that can be checked off a list; rather, they are highly interrelated tools that should be utilized frequently and in diverse contexts.

The sample items included with this curriculum framework demonstrate various ways in which the learning objectives can be linked with the Mathematical Practices for AP Calculus.

The Mathematical Practices for AP Calculus are given below.

### MPAC 1: Reasoning with definitions and theorems

Students can:

- ▶ use definitions and theorems to build arguments, to justify conclusions or answers, and to prove results;
- ▶ confirm that hypotheses have been satisfied in order to apply the conclusion of a theorem;
- ▶ apply definitions and theorems in the process of solving a problem;
- ▶ interpret quantifiers in definitions and theorems (e.g., “for all,” “there exists”);
- ▶ develop conjectures based on exploration with technology; and
- ▶ produce examples and counterexamples to clarify understanding of definitions, to investigate whether converses of theorems are true or false, or to test conjectures.

### MPAC 2: Connecting concepts

Students can:

- ▶ relate the concept of a limit to all aspects of calculus;
- ▶ use the connection between concepts (e.g., rate of change and accumulation) or processes (e.g., differentiation and its inverse process, antidifferentiation) to solve problems;

- ▶ connect concepts to their visual representations with and without technology; and
- ▶ identify a common underlying structure in problems involving different contextual situations.

### **MPAC 3: Implementing algebraic/computational processes**

Students can:

- ▶ select appropriate mathematical strategies;
- ▶ sequence algebraic/computational procedures logically;
- ▶ complete algebraic/computational processes correctly;
- ▶ apply technology strategically to solve problems;
- ▶ attend to precision graphically, numerically, analytically, and verbally and specify units of measure; and
- ▶ connect the results of algebraic/computational processes to the question asked.

### **MPAC 4: Connecting multiple representations**

Students can:

- ▶ associate tables, graphs, and symbolic representations of functions;
- ▶ develop concepts using graphical, symbolical, or numerical representations with and without technology;
- ▶ identify how mathematical characteristics of functions are related in different representations;
- ▶ extract and interpret mathematical content from any presentation of a function (e.g., utilize information from a table of values);
- ▶ construct one representational form from another (e.g., a table from a graph or a graph from given information); and
- ▶ consider multiple representations of a function to select or construct a useful representation for solving a problem.

### **MPAC 5: Building notational fluency**

Students can:

- ▶ know and use a variety of notations (e.g.,  $f'(x)$ ,  $y'$ ,  $\frac{dy}{dx}$ );
- ▶ connect notation to definitions (e.g., relating the notation for the definite integral to that of the limit of a Riemann sum);
- ▶ connect notation to different representations (graphical, numerical, analytical, and verbal); and
- ▶ assign meaning to notation, accurately interpreting the notation in a given problem and across different contexts.

## **MPAC 6: Communicating**

Students can:

- ▶ clearly present methods, reasoning, justifications, and conclusions;
- ▶ use accurate and precise language and notation;
- ▶ explain the meaning of expressions, notation, and results in terms of a context (including units);
- ▶ explain the connections among concepts;
- ▶ critically interpret and accurately report information provided by technology; and
- ▶ analyze, evaluate, and compare the reasoning of others.

## Use of Graphing Calculators and Other Technologies in AP Calculus

The use of a graphing calculator is considered an integral part of the AP Calculus courses, and it is required on some portions of the exams. Professional mathematics organizations such as the National Council of Teachers of Mathematics (NCTM), the Mathematical Association of America (MAA), and the National Academy of Sciences (NAS) Board on Mathematical Sciences and Their Applications have strongly endorsed the use of calculators in mathematics instruction and testing.

Graphing calculators are valuable tools for achieving multiple components of the Mathematical Practices for AP Calculus, including using technology to develop conjectures, connecting concepts to their visual representations, solving problems, and critically interpreting and accurately reporting information. Appropriate examples of graphing calculator use in AP Calculus include, but certainly are not limited to, zooming to reveal local linearity, constructing a table of values to conjecture a limit, developing a visual representation of Riemann sums approaching a definite integral, graphing Taylor polynomials to understand intervals of convergence for Taylor series, and drawing a slope field and investigating how the choice of initial condition affects the solution to a differential equation.

The AP Calculus Program also supports the use of other technologies that are available to students and encourages teachers to incorporate technology into instruction in a variety of ways as a means of facilitating discovery and reflection.

# The Concept Outline

## Big Idea 1: Limits

Many calculus concepts are developed by first considering a discrete model and then the consequences of a limiting case. Therefore, the idea of limits is essential for discovering and developing important ideas, definitions, formulas, and theorems in calculus. Students must have a solid, intuitive understanding of limits and be able to compute various limits, including one-sided limits, limits at infinity, the limit of a sequence, and infinite limits. They should be able to work with tables and graphs in order to estimate the limit of a function at a point. Students should know the algebraic properties of limits and techniques for finding limits of indeterminate forms, and they should be able to apply limits to understand the behavior of a function near a point. Students must also understand how limits are used to determine continuity, a fundamental property of functions.

### Enduring Understandings

(Students will understand that . . . )

**EU 1.1:** The concept of a limit can be used to understand the behavior of functions.

### Learning Objectives

(Students will be able to . . . )

**LO 1.1A(a):** Express limits symbolically using correct notation.

**LO 1.1A(b):** Interpret limits expressed symbolically.

**LO 1.1B:** Estimate limits of functions.

### Essential Knowledge

(Students will know that . . . )

**EK 1.1A1:** Given a function  $f$ , the limit of  $f(x)$  as  $x$  approaches  $c$  is a real number  $R$  if  $f(x)$  can be made arbitrarily close to  $R$  by taking  $x$  sufficiently close to  $c$  (but not equal to  $c$ ). If the limit exists and is a real number, then the common notation is  $\lim_{x \rightarrow c} f(x) = R$ .

#### EXCLUSION STATEMENT (EK 1.1A1):

*The epsilon-delta definition of a limit is not assessed on the AP Calculus AB or BC Exam. However, teachers may include this topic in the course if time permits.*

**EK 1.1A2:** The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.

**EK 1.1A3:** A limit might not exist for some functions at particular values of  $x$ . Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

#### EXAMPLES OF LIMITS THAT DO NOT EXIST:

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$$

$$\lim_{x \rightarrow 0} \frac{|x|}{x} \text{ does not exist}$$

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) \text{ does not exist}$$

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ does not exist}$$

**EK 1.1B1:** Numerical and graphical information can be used to estimate limits.

**Note:** In the Concept Outline, subject matter that is included only in the BC course is indicated with blue shading.

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 1.1:</b> The concept of a limit can be used to understand the behavior of functions.</p> <p><i>(continued)</i></p>	<p><b>LO 1.1C:</b> Determine limits of functions.</p>	<p><b>EK 1.1C1:</b> Limits of sums, differences, products, quotients, and composite functions can be found using the basic theorems of limits and algebraic rules.</p> <p><b>EK 1.1C2:</b> The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem.</p> <p><b>EK 1.1C3:</b> Limits of the indeterminate forms <math>\frac{0}{0}</math> and <math>\frac{\infty}{\infty}</math> may be evaluated using L'Hospital's Rule.</p>
<p><b>EU 1.2:</b> Continuity is a key property of functions that is defined using limits.</p>	<p><b>LO 1.1D:</b> Deduce and interpret behavior of functions using limits.</p>	<p><b>EK 1.1D1:</b> Asymptotic and unbounded behavior of functions can be explained and described using limits.</p> <p><b>EK 1.1D2:</b> Relative magnitudes of functions and their rates of change can be compared using limits.</p>
<p><b>EU 1.2:</b> Continuity is a key property of functions that is defined using limits.</p>	<p><b>LO 1.2A:</b> Analyze functions for intervals of continuity or points of discontinuity.</p>	<p><b>EK 1.2A1:</b> A function <math>f</math> is continuous at <math>x = c</math> provided that <math>f(c)</math> exists, <math>\lim_{x \rightarrow c} f(x)</math> exists, and <math>\lim_{x \rightarrow c} f(x) = f(c)</math>.</p> <p><b>EK 1.2A2:</b> Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous at all points in their domains.</p> <p><b>EK 1.2A3:</b> Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.</p>
<p><b>EU 1.2:</b> Continuity is a key property of functions that is defined using limits.</p>	<p><b>LO 1.2B:</b> Determine the applicability of important calculus theorems using continuity.</p>	<p><b>EK 1.2B1:</b> Continuity is an essential condition for theorems such as the Intermediate Value Theorem, the Extreme Value Theorem, and the Mean Value Theorem.</p>

## Big Idea 2: Derivatives

Using derivatives to describe the rate of change of one variable with respect to another variable allows students to understand change in a variety of contexts. In AP Calculus, students build the derivative using the concept of limits and use the derivative primarily to compute the instantaneous rate of change of a function. Applications of the derivative include finding the slope of a tangent line to a graph at a point, analyzing the graph of a function (for example, determining whether a function is increasing or decreasing and finding concavity and extreme values), and solving problems involving rectilinear motion. Students should be able to use different definitions of the derivative, estimate derivatives from tables and graphs, and apply various derivative rules and properties. In addition, students should be able to solve separable differential equations, understand and be able to apply the Mean Value Theorem, and be familiar with a variety of real-world applications, including related rates, optimization, and growth and decay models.

### Enduring Understandings

(Students will understand that . . . )

**EU 2.1:** The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies.

### Learning Objectives

(Students will be able to . . . )

**LO 2.1A:** Identify the derivative of a function as the limit of a difference quotient.

**LO 2.1B:** Estimate derivatives.

### Essential Knowledge

(Students will know that . . . )

**EK 2.1A1:** The difference quotients  $\frac{f(a+h)-f(a)}{h}$  and  $\frac{f(x)-f(a)}{x-a}$  express the average rate of change of a function over an interval.

**EK 2.1A2:** The instantaneous rate of change of a function at a point can be expressed by  $\lim_{h \rightarrow 0} \frac{f(a+h)-f(a)}{h}$  or  $\lim_{x \rightarrow a} \frac{f(x)-f(a)}{x-a}$ , provided that the limit exists. These are common forms of the definition of the derivative and are denoted  $f'(a)$ .

**EK 2.1A3:** The derivative of  $f$  is the function whose value at  $x$  is  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$  provided this limit exists.

**EK 2.1A4:** For  $y = f(x)$ , notations for the derivative include  $\frac{dy}{dx}$ ,  $f'(x)$ , and  $y'$ .

**EK 2.1A5:** The derivative can be represented graphically, numerically, analytically, and verbally.

**EK 2.1B1:** The derivative at a point can be estimated from information given in tables or graphs.



<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 2.1:</b> The derivative of a function is defined as the limit of a difference quotient and can be determined using a variety of strategies.</p> <p><i>(continued)</i></p>	<p><b>LO 2.1C:</b> Calculate derivatives.</p>	<p><b>EK 2.1C1:</b> Direct application of the definition of the derivative can be used to find the derivative for selected functions, including polynomial, power, sine, cosine, exponential, and logarithmic functions.</p> <p><b>EK 2.1C2:</b> Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric.</p> <p><b>EK 2.1C3:</b> Sums, differences, products, and quotients of functions can be differentiated using derivative rules.</p> <p><b>EK 2.1C4:</b> The chain rule provides a way to differentiate composite functions.</p> <p><b>EK 2.1C5:</b> The chain rule is the basis for implicit differentiation.</p> <p><b>EK 2.1C6:</b> The chain rule can be used to find the derivative of an inverse function, provided the derivative of that function exists.</p> <p><b>EK 2.1C7: (BC)</b> Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.</p>
<p><b>EU 2.2:</b> A function's derivative, which is itself a function, can be used to understand the behavior of the function.</p>	<p><b>LO 2.2A:</b> Use derivatives to analyze properties of a function.</p>	<p><b>EK 2.1D1:</b> Differentiating <math>f'</math> produces the second derivative <math>f''</math>, provided the derivative of <math>f'</math> exists; repeating this process produces higher order derivatives of <math>f</math>.</p> <p><b>EK 2.1D2:</b> Higher order derivatives are represented with a variety of notations. For <math>y = f(x)</math>, notations for the second derivative include <math>\frac{d^2y}{dx^2}</math>, <math>f''(x)</math>, and <math>y''</math>. Higher order derivatives can be denoted <math>\frac{d^n y}{dx^n}</math> or <math>f^{(n)}(x)</math>.</p> <p><b>EK 2.2A1:</b> First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.</p>

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 2.2:</b> A function's derivative, which is itself a function, can be used to understand the behavior of the function.</p> <p><i>(continued)</i></p>	<p><b>LO 2.2A:</b> Use derivatives to analyze properties of a function.</p> <p><i>(continued)</i></p>	<p><b>EK 2.2A2:</b> Key features of functions and their derivatives can be identified and related to their graphical, numerical, and analytical representations.</p> <p><b>EK 2.2A3:</b> Key features of the graphs of <math>f</math>, <math>f'</math>, and <math>f''</math> are related to one another.</p> <p><b>EK 2.2A4: (BC)</b> For a curve given by a polar equation <math>r = f(\theta)</math>, derivatives of <math>r</math>, <math>x</math>, and <math>y</math> with respect to <math>\theta</math> and first and second derivatives of <math>y</math> with respect to <math>x</math> can provide information about the curve.</p>
<p><b>EU 2.3:</b> The derivative has multiple interpretations and applications including those that involve instantaneous rates of change.</p>	<p><b>LO 2.2B:</b> Recognize the connection between differentiability and continuity.</p>	<p><b>EK 2.2B1:</b> A continuous function may fail to be differentiable at a point in its domain.</p> <p><b>EK 2.2B2:</b> If a function is differentiable at a point, then it is continuous at that point.</p>
	<p><b>LO 2.3A:</b> Interpret the meaning of a derivative within a problem.</p>	<p><b>EK 2.3A1:</b> The unit for <math>f'(x)</math> is the unit for <math>f</math> divided by the unit for <math>x</math>.</p> <p><b>EK 2.3A2:</b> The derivative of a function can be interpreted as the instantaneous rate of change with respect to its independent variable.</p>
	<p><b>LO 2.3B:</b> Solve problems involving the slope of a tangent line.</p>	<p><b>EK 2.3B1:</b> The derivative at a point is the slope of the line tangent to a graph at that point on the graph.</p> <p><b>EK 2.3B2:</b> The tangent line is the graph of a locally linear approximation of the function near the point of tangency.</p>
	<p><b>LO 2.3C:</b> Solve problems involving related rates, optimization, rectilinear motion, <b>(BC)</b> and planar motion.</p>	<p><b>EK 2.3C1:</b> The derivative can be used to solve rectilinear motion problems involving position, speed, velocity, and acceleration.</p> <p><b>EK 2.3C2:</b> The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.</p> <p><b>EK 2.3C3:</b> The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval.</p> <p><b>EK 2.3C4: (BC)</b> Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along curves given by parametric or vector-valued functions.</p>

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 2.3:</b> The derivative has multiple interpretations and applications including those that involve instantaneous rates of change.</p> <p><i>(continued)</i></p>	<p><b>LO 2.3D:</b> Solve problems involving rates of change in applied contexts.</p>	<p><b>EK 2.3D1:</b> The derivative can be used to express information about rates of change in applied contexts.</p>
	<p><b>LO 2.3E:</b> Verify solutions to differential equations.</p>	<p><b>EK 2.3E1:</b> Solutions to differential equations are functions or families of functions.</p>
	<p><b>LO 2.3F:</b> Estimate solutions to differential equations.</p>	<p><b>EK 2.3E2:</b> Derivatives can be used to verify that a function is a solution to a given differential equation.</p>
		<p><b>EK 2.3F1:</b> Slope fields provide visual clues to the behavior of solutions to first order differential equations.</p>
		<p><b>EK 2.3F2: (BC)</b> For differential equations, Euler’s method provides a procedure for approximating a solution or a point on a solution curve.</p>
<p><b>EU 2.4:</b> The Mean Value Theorem connects the behavior of a differentiable function over an interval to the behavior of the derivative of that function at a particular point in the interval.</p>	<p><b>LO 2.4A:</b> Apply the Mean Value Theorem to describe the behavior of a function over an interval.</p>	<p><b>EK 2.4A1:</b> If a function <math>f</math> is continuous over the interval <math>[a, b]</math> and differentiable over the interval <math>(a, b)</math>, the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.</p>

## Big Idea 3: Integrals and the Fundamental Theorem of Calculus

Integrals are used in a wide variety of practical and theoretical applications. AP Calculus students should understand the definition of a definite integral involving a Riemann sum, be able to approximate a definite integral using different methods, and be able to compute definite integrals using geometry. They should be familiar with basic techniques of integration and properties of integrals. The interpretation of a definite integral is an important skill, and students should be familiar with area, volume, and motion applications, as well as with the use of the definite integral as an accumulation function. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus — a central idea in AP Calculus. Students should be able to work with and analyze functions defined by an integral.

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<b>EU 3.1:</b> Antidifferentiation is the inverse process of differentiation.	<b>LO 3.1A:</b> Recognize antiderivatives of basic functions.	<b>EK 3.1A1:</b> An antiderivative of a function $f$ is a function $g$ whose derivative is $f$ . <b>EK 3.1A2:</b> Differentiation rules provide the foundation for finding antiderivatives.
<b>EU 3.2:</b> The definite integral of a function over an interval is the limit of a Riemann sum over that interval and can be calculated using a variety of strategies.	<b>LO 3.2A(a):</b> Interpret the definite integral as the limit of a Riemann sum. <b>LO 3.2A(b):</b> Express the limit of a Riemann sum in integral notation.	<b>EK 3.2A1:</b> A Riemann sum, which requires a partition of an interval $I$ , is the sum of products, each of which is the value of the function at a point in a subinterval multiplied by the length of that subinterval of the partition. <b>EK 3.2A2:</b> The definite integral of a continuous function $f$ over the interval $[a, b]$ , denoted by $\int_a^b f(x)dx$ , is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$ where $x_i^*$ is a value in the $i$ th subinterval, $\Delta x_i$ is the width of the $i$ th subinterval, $n$ is the number of subintervals, and $\max \Delta x_i$ is the width of the largest subinterval. Another form of the definition is $\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x_i$ , where $\Delta x_i = \frac{b-a}{n}$ and $x_i^*$ is a value in the $i$ th subinterval. <b>EK 3.2A3:</b> The information in a definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.

**Enduring Understandings**

(Students will understand that . . . )

**Learning Objectives**

(Students will be able to . . . )

**Essential Knowledge**

(Students will know that . . . )

**EU 3.2:** The definite integral of a function over an interval is the limit of a Riemann sum over that interval and can be calculated using a variety of strategies.

*(continued)*

**LO 3.2B:** Approximate a definite integral.

**EK 3.2B1:** Definite integrals can be approximated for functions that are represented graphically, numerically, algebraically, and verbally.

**EK 3.2B2:** Definite integrals can be approximated using a left Riemann sum, a right Riemann sum, a midpoint Riemann sum, or a trapezoidal sum; approximations can be computed using either uniform or nonuniform partitions.

**LO 3.2C:** Calculate a definite integral using areas and properties of definite integrals.

**EK 3.2C1:** In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.

**EK 3.2C2:** Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.

**EK 3.2C3:** The definition of the definite integral may be extended to functions with removable or jump discontinuities.

**LO 3.2D: (BC)** Evaluate an improper integral or show that an improper integral diverges.

**EK 3.2D1: (BC)** An improper integral is an integral that has one or both limits infinite or has an integrand that is unbounded in the interval of integration.

**EK 3.2D2: (BC)** Improper integrals can be determined using limits of definite integrals.

**EU 3.3:** The Fundamental Theorem of Calculus, which has two distinct formulations, connects differentiation and integration.

**LO 3.3A:** Analyze functions defined by an integral.

**EK 3.3A1:** The definite integral can be used to define new functions; for example,  $f(x) = \int_0^x e^{-t^2} dt$ .

**EK 3.3A2:** If  $f$  is a continuous function on the interval  $[a, b]$ , then  $\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$ , where  $x$  is between  $a$  and  $b$ .

**EK 3.3A3:** Graphical, numerical, analytical, and verbal representations of a function  $f$  provide information about the function  $g$  defined as  $g(x) = \int_a^x f(t) dt$ .

**LO 3.3B(a):** Calculate antiderivatives.

**EK 3.3B1:** The function defined by  $F(x) = \int_a^x f(t) dt$  is an antiderivative of  $f$ .

**LO 3.3B(b):** Evaluate definite integrals.

**EK 3.3B2:** If  $f$  is continuous on the interval  $[a, b]$  and  $F$  is an antiderivative of  $f$ , then  $\int_a^b f(x) dx = F(b) - F(a)$ .

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 3.3:</b> The Fundamental Theorem of Calculus, which has two distinct formulations, connects differentiation and integration.</p> <p><i>(continued)</i></p>	<p><b>LO 3.3B(a):</b> Calculate antiderivatives.</p> <p><b>LO 3.3B(b):</b> Evaluate definite integrals.</p> <p><i>(continued)</i></p>	<p><b>EK 3.3B3:</b> The notation <math>\int f(x)dx = F(x) + C</math> means that <math>F'(x) = f(x)</math>, and <math>\int f(x)dx</math> is called an indefinite integral of the function <math>f</math>.</p> <p><b>EK 3.3B4:</b> Many functions do not have closed form antiderivatives.</p> <p><b>EK 3.3B5:</b> Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, <b>(BC)</b> integration by parts, and nonrepeating linear partial fractions.</p>
<p><b>EU 3.4:</b> The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation.</p>	<p><b>LO 3.4A:</b> Interpret the meaning of a definite integral within a problem.</p>	<p><b>EK 3.4A1:</b> A function defined as an integral represents an accumulation of a rate of change.</p> <p><b>EK 3.4A2:</b> The definite integral of the rate of change of a quantity over an interval gives the net change of that quantity over that interval.</p> <p><b>EK 3.4A3:</b> The limit of an approximating Riemann sum can be interpreted as a definite integral.</p>
<p><b>LO 3.4B:</b> Apply definite integrals to problems involving the average value of a function.</p>	<p><b>LO 3.4B:</b> Apply definite integrals to problems involving the average value of a function.</p>	<p><b>EK 3.4B1:</b> The average value of a function <math>f</math> over an interval <math>[a, b]</math> is <math>\frac{1}{b-a} \int_a^b f(x)dx</math>.</p>
<p><b>LO 3.4C:</b> Apply definite integrals to problems involving motion.</p>	<p><b>LO 3.4C:</b> Apply definite integrals to problems involving motion.</p>	<p><b>EK 3.4C1:</b> For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.</p> <p><b>EK 3.4C2: (BC)</b> The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.</p>

**Enduring Understandings**

(Students will understand that . . . )

**EU 3.4:** The definite integral of a function over an interval is a mathematical tool with many interpretations and applications involving accumulation.

*(continued)*

**EU 3.5:** Antidifferentiation is an underlying concept involved in solving separable differential equations. Solving separable differential equations involves determining a function or relation given its rate of change.

**Learning Objectives**

(Students will be able to . . . )

**LO 3.4D:** Apply definite integrals to problems involving area, volume, **(BC)** and length of a curve.

**LO 3.4E:** Use the definite integral to solve problems in various contexts.

**LO 3.5A:** Analyze differential equations to obtain general and specific solutions.

**LO 3.5B:** Interpret, create and solve differential equations from problems in context.

**Essential Knowledge**

(Students will know that . . . )

**EK 3.4D1:** Areas of certain regions in the plane can be calculated with definite integrals. **(BC)** Areas bounded by polar curves can be calculated with definite integrals.

**EK 3.4D2:** Volumes of solids with known cross sections, including discs and washers, can be calculated with definite integrals.

**EK 3.4D3: (BC)** The length of a planar curve defined by a function or by a parametrically defined curve can be calculated using a definite integral.

**EK 3.4E1:** The definite integral can be used to express information about accumulation and net change in many applied contexts.

**EK 3.5A1:** Antidifferentiation can be used to find specific solutions to differential equations with given initial conditions, including applications to motion along a line, exponential growth and decay, **(BC)** and logistic growth.

**EK 3.5A2:** Some differential equations can be solved by separation of variables.

**EK 3.5A3:** Solutions to differential equations may be subject to domain restrictions.

**EK 3.5A4:** The function  $F$  defined by  $F(x) = c + \int_a^x f(t)dt$  is a general solution to the differential equation  $\frac{dy}{dx} = f(x)$ , and  $F(x) = y_0 + \int_a^x f(t)dt$  is a particular solution to the differential equation  $\frac{dy}{dx} = f(x)$  satisfying  $F(a) = y_0$ .

**EK 3.5B1:** The model for exponential growth and decay that arises from the statement “The rate of change of a quantity is proportional to the size of the quantity” is  $\frac{dy}{dt} = ky$ .

**EK 3.5B2: (BC)** The model for logistic growth that arises from the statement “The rate of change of a quantity is jointly proportional to the size of the quantity and the difference between the quantity and the carrying capacity” is  $\frac{dy}{dt} = ky(a - y)$ .

## Big Idea 4: Series (BC)

The AP Calculus BC curriculum includes the study of series of numbers, power series, and various methods to determine convergence or divergence of a series. Students should be familiar with Maclaurin series for common functions and general Taylor series representations. Other topics include the radius and interval of convergence and operations on power series. The technique of using power series to approximate an arbitrary function near a specific value allows for an important connection to the tangent-line problem and is a natural extension that helps achieve a better approximation. The concept of approximation is a common theme throughout AP Calculus, and power series provide a unifying, comprehensive conclusion.

### Enduring

#### Understandings

(Students will understand that . . . )

#### Learning Objectives

(Students will be able to . . . )

#### Essential Knowledge

(Students will know that . . . )

**EU 4.1:** The sum of an infinite number of real numbers may converge.

**LO 4.1A:** Determine whether a series converges or diverges.

**EK 4.1A1:** The  $n$ th partial sum is defined as the sum of the first  $n$  terms of a sequence.

**EK 4.1A2:** An infinite series of numbers converges to a real number  $S$  (or has sum  $S$ ), if and only if the limit of its sequence of partial sums exists and equals  $S$ .

**EK 4.1A3:** Common series of numbers include geometric series, the harmonic series, and  $p$ -series.

**EK 4.1A4:** A series may be absolutely convergent, conditionally convergent, or divergent.

**EK 4.1A5:** If a series converges absolutely, then it converges.

**EK 4.1A6:** In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the  $n$ th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.

#### EXCLUSION STATEMENT (EK 4.1A6):

*Other methods for determining convergence or divergence of a series of numbers are not assessed on the AP Calculus AB or BC Exam. However, teachers may include these topics in the course if time permits.*



**Enduring Understandings**

(Students will understand that . . . )

**Learning Objectives**

(Students will be able to . . . )

**Essential Knowledge**

(Students will know that . . . )

**EU 4.1:** The sum of an infinite number of real numbers may converge.

*(continued)*

**LO 4.1B:** Determine or estimate the sum of a series.

**EK 4.1B1:** If  $a$  is a real number and  $r$  is a real number such that  $|r| < 1$ , then the geometric series  $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ .

**EK 4.1B2:** If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.

**EK 4.1B3:** If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value.

**EU 4.2:** A function can be represented by an associated power series over the interval of convergence for the power series.

**LO 4.2A:** Construct and use Taylor polynomials.

**EK 4.2A1:** The coefficient of the  $n$ th-degree term in a Taylor polynomial centered at  $x = a$  for the function  $f$  is  $\frac{f^{(n)}(a)}{n!}$ .

**EK 4.2A2:** Taylor polynomials for a function  $f$  centered at  $x = a$  can be used to approximate function values of  $f$  near  $x = a$ .

**EK 4.2A3:** In many cases, as the degree of a Taylor polynomial increases, the  $n$ th-degree polynomial will converge to the original function over some interval.

**EK 4.2A4:** The Lagrange error bound can be used to bound the error of a Taylor polynomial approximation to a function.

**EK 4.2A5:** In some situations where the signs of a Taylor polynomial are alternating, the alternating series error bound can be used to bound the error of a Taylor polynomial approximation to the function.

**LO 4.2B:** Write a power series representing a given function.

**EK 4.2B1:** A power series is a series of the form  $\sum_{n=0}^{\infty} a_n(x-r)^n$  where  $n$  is a non-negative integer,  $\{a_n\}$  is a sequence of real numbers, and  $r$  is a real number.

**EK 4.2B2:** The Maclaurin series for  $\sin(x)$ ,  $\cos(x)$ , and  $e^x$  provide the foundation for constructing the Maclaurin series for other functions.

**EK 4.2B3:** The Maclaurin series for  $\frac{1}{1-x}$  is a geometric series.

**EK 4.2B4:** A Taylor polynomial for  $f(x)$  is a partial sum of the Taylor series for  $f(x)$ .

<b>Enduring Understandings</b> (Students will understand that . . . )	<b>Learning Objectives</b> (Students will be able to . . . )	<b>Essential Knowledge</b> (Students will know that . . . )
<p><b>EU 4.2:</b> A function can be represented by an associated power series over the interval of convergence for the power series.</p> <p><i>(continued)</i></p>	<p><b>LO 4.2B:</b> Write a power series representing a given function.</p> <p><i>(continued)</i></p> <hr/> <p><b>LO 4.2C:</b> Determine the radius and interval of convergence of a power series.</p>	<p><b>EK 4.2B5:</b> A power series for a given function can be derived by various methods (e.g., algebraic processes, substitutions, using properties of geometric series, and operations on known series such as term-by-term integration or term-by-term differentiation).</p> <hr/> <p><b>EK 4.2C1:</b> If a power series converges, it either converges at a single point or has an interval of convergence.</p> <p><b>EK 4.2C2:</b> The ratio test can be used to determine the radius of convergence of a power series.</p> <hr/> <p><b>EK 4.2C3:</b> If a power series has a positive radius of convergence, then the power series is the Taylor series of the function to which it converges over the open interval.</p> <hr/> <p><b>EK 4.2C4:</b> The radius of convergence of a power series obtained by term-by-term differentiation or term-by-term integration is the same as the radius of convergence of the original power series.</p>

# The AP Calculus Exams

## Exam Information

Students take either the AP Calculus AB Exam or the AP Calculus BC Exam. The exams, which are identical in format, consist of a multiple-choice section and a free-response section, as shown below.

### Section I: Multiple Choice

Part	Graphing Calculator	Number of Questions	Time
Part A	Not permitted	30	60 minutes
Part B	Required	15	45 minutes
<b>TOTAL</b>		<b>45</b>	<b>105 minutes</b>

### Section II: Free Response

Part	Graphing Calculator	Number of Questions	Time
Part A	Required	2	30 minutes
Part B	Not permitted	4	60 minutes
<b>TOTAL</b>		<b>6</b>	<b>90 minutes</b>

Student performance on these two sections will be compiled and weighted to determine an AP Exam score. Each section of the exam counts toward 50 percent of the student's score. Points are not deducted for incorrect answers or unanswered questions.

The learning objectives are the targets for assessment. Exam questions assess the learning objectives detailed in the concept outline; as such, they require a strong conceptual understanding of calculus in conjunction with the application of one or more of the Mathematical Practices for AP Calculus. Although topics in subject areas such as algebra, geometry, and precalculus are not explicitly assessed, students must have mastered the relevant preparatory material in order to apply calculus techniques successfully and accurately.

The multiple-choice section on each exam is designed for broad coverage of the course content. Multiple-choice questions are discrete, as opposed to appearing in question sets, and the questions do not appear in the order in which topics are addressed in the curriculum framework. Each part of the multiple-choice section is timed. Students may not return to questions in Part A of the multiple-choice section once they have begun Part B.

Free-response questions provide students with an opportunity to demonstrate their knowledge of correct mathematical reasoning and thinking. In most cases, an answer without supporting work will receive no credit; students are required to articulate the reasoning and methods that support their answer. Some questions will ask students to justify an answer or discuss whether a theorem can be applied. Each part of the free-response section is timed, and students may use a graphing calculator only for Part A. During the timed portion for Part B of the free-response section, students are allowed to return to working on Part A questions, though without the use of a graphing calculator.

### **Calculus AB Subscore for the Calculus BC Exam**

Common topics are assessed at the same conceptual level on both of the AP Calculus Exams. Students who take the AP Calculus BC Exam receive an AP Calculus AB subscore based on their performance on the portion of the exam devoted to Calculus AB topics (approximately 60 percent of the exam). The Calculus AB subscore is designed to give students as well as colleges and universities more information about the achievement of AP Calculus BC students.

### **Calculator Use on the Exams**

Both the multiple-choice and free-response sections of the AP Calculus Exams include problems that require the use of a graphing calculator. A graphing calculator appropriate for use on the exams is expected to have the built-in capability to do the following:

- ▶ Plot the graph of a function within an arbitrary viewing window
- ▶ Find the zeros of functions (solve equations numerically)
- ▶ Numerically calculate the derivative of a function
- ▶ Numerically calculate the value of a definite integral

One or more of these capabilities should provide sufficient computational tools for successful development of a solution to any AP Calculus AB or BC Exam question that requires the use of a calculator. Care is taken to ensure that the exam questions do not favor students who use graphing calculators with more extensive built-in features.

Students are expected to bring a graphing calculator with the capabilities listed above to the exams. AP teachers should check their own students' calculators to ensure that the required conditions are met. Students and teachers should keep their calculators updated with the latest available operating systems. Information is available on calculator company websites. A list of acceptable calculators can be found at the AP Central® website (<http://apcentral.collegeboard.com>).

Note that requirements regarding calculator use help ensure that all students have sufficient computational tools for the AP Calculus Exams. Exam restrictions should not be interpreted as restrictions on classroom activities.

## Sample Exam Questions

The sample questions that follow illustrate the relationship between the *AP Calculus AB and AP Calculus BC Curriculum Framework* and the redesigned AP Calculus Exams and serve as examples of the types of questions that will appear on the exams. Sample questions addressing the new content of the courses have been deliberately included; as such, the topic distribution of these questions is not indicative of the distribution on the actual exam.

Each question is accompanied by a table containing the main learning objective(s), essential knowledge statement(s), and Mathematical Practice(s) for AP Calculus that the question addresses. In addition, each free-response question is accompanied by an explanation of how the relevant Mathematical Practices for AP Calculus can be applied in answering the question. The information accompanying each question is intended to aid in identifying the focus of the question, with the underlying assumption that learning objectives, essential knowledge statements, and MPACs other than those listed may also partially apply. Note that in the cases where multiple learning objectives, essential knowledge statements, or MPACs are provided for a multiple-choice question, the primary one is listed first.

## AP Calculus AB Sample Exam Questions

### Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.

1.  $\lim_{x \rightarrow \pi} \frac{\cos x + \sin(2x) + 1}{x^2 - \pi^2}$  is

(A)  $\frac{1}{2\pi}$

(B)  $\frac{1}{\pi}$

(C) 1

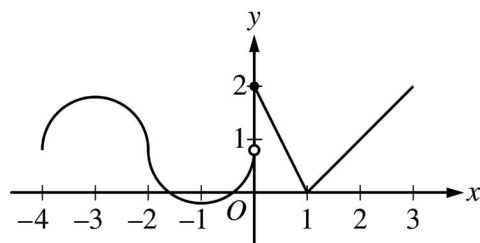
(D) nonexistent

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 1.1C:</b> Determine limits of functions.	<b>EK 1.1C3:</b> Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.	<b>MPAC 1:</b> Reasoning with definitions and theorems
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C2:</b> Specific rules can be used to calculate derivatives for classes of functions, including polynomial, rational, power, exponential, logarithmic, trigonometric, and inverse trigonometric.	<b>MPAC 3:</b> Implementing algebraic/computational processes

2.  $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^4 + 1}}{x^2 - 3x + 5}$  is

- (A) 1  
 (B) 3  
 (C) 9  
 (D) nonexistent

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 1.1C:</b> Determine limits of functions.	<b>EK 1.1C2:</b> The limit of a function may be found by using algebraic manipulation, alternate forms of trigonometric functions, or the squeeze theorem.	<b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 1.1A(b):</b> Interpret limits expressed symbolically.	<b>EK 1.1A2:</b> The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	<b>MPAC 2:</b> Connecting concepts

Graph of  $f$ 

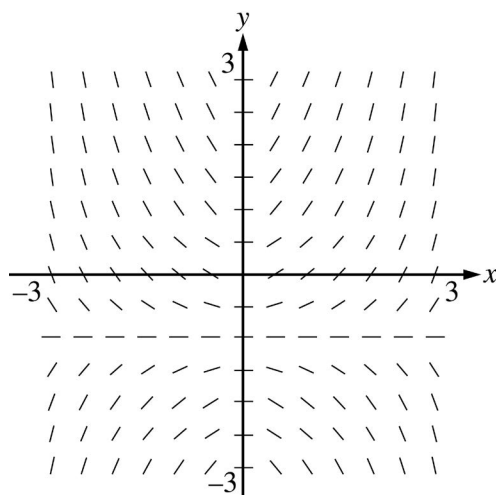
3. The graph of the piecewise-defined function  $f$  is shown in the figure above. The graph has a vertical tangent line at  $x = -2$  and horizontal tangent lines at  $x = -3$  and  $x = -1$ . What are all values of  $x$ ,  $-4 < x < 3$ , at which  $f$  is continuous but not differentiable?
- (A)  $x = 1$
- (B)  $x = -2$  and  $x = 0$
- (C)  $x = -2$  and  $x = 1$
- (D)  $x = 0$  and  $x = 1$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.2B:</b> Recognize the connection between differentiability and continuity.	<b>EK 2.2B1:</b> A continuous function may fail to be differentiable at a point in its domain.	<b>MPAC 4:</b> Connecting multiple representations
<b>LO 1.2A:</b> Analyze functions for intervals of continuity or points of discontinuity.	<b>EK 1.2A3:</b> Types of discontinuities include removable discontinuities, jump discontinuities, and discontinuities due to vertical asymptotes.	<b>MPAC 2:</b> Connecting concepts



4. An ice sculpture in the form of a sphere melts in such a way that it maintains its spherical shape. The volume of the sphere is decreasing at a constant rate of  $2\pi$  cubic meters per hour. At what rate, in square meters per hour, is the surface area of the sphere decreasing at the moment when the radius is 5 meters? (Note: For a sphere of radius  $r$ , the surface area is  $4\pi r^2$  and the volume is  $\frac{4}{3}\pi r^3$ .)
- (A)  $\frac{4\pi}{5}$
- (B)  $40\pi$
- (C)  $80\pi^2$
- (D)  $100\pi$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.3C:</b> Solve problems involving related rates, optimization, rectilinear motion, (BC) and planar motion.	<b>EK 2.3C2:</b> The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.	<b>MPAC 2:</b> Connecting concepts <b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C5:</b> The chain rule is the basis for implicit differentiation.	



5. Shown above is a slope field for which of the following differential equations?

(A)  $\frac{dy}{dx} = xy + x$

(B)  $\frac{dy}{dx} = xy + y$

(C)  $\frac{dy}{dx} = y + 1$

(D)  $\frac{dy}{dx} = (x + 1)^2$

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**Learning Objective**

**LO 2.3F:** Estimate solutions to differential equations.

**Essential Knowledge**

**EK 2.3F1:** Slope fields provide visual clues to the behavior of solutions to first order differential equations.

**Mathematical Practices for AP Calculus**

**MPAC 4:** Connecting multiple representations

**MPAC 2:** Connecting concepts

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$$f(x) = \begin{cases} 2x - 2 & \text{for } x < 3 \\ 2x - 4 & \text{for } x \geq 3 \end{cases}$$

6. Let  $f$  be the piecewise-linear function defined above. Which of the following statements are true?

I.  $\lim_{h \rightarrow 0^-} \frac{f(3+h) - f(3)}{h} = 2$

II.  $\lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{h} = 2$

III.  $f'(3) = 2$

- (A) None  
 (B) II only  
 (C) I and II only  
 (D) I, II, and III

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.1A:</b> Identify the derivative of a function as the limit of a difference quotient.	<b>EK 2.1A2:</b> The instantaneous rate of change of a function at a point can be expressed by  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ or  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a},$ provided that the limit exists. These are common forms of the definition of the derivative and are denoted $f'(a)$ .	<b>MPAC 2:</b> Connecting concepts  <b>MPAC 5:</b> Building notational fluency
<b>LO 1.1A(b):</b> Interpret limits expressed symbolically.	<b>EK 1.1A2:</b> The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	

7. If  $f(x) = \int_1^{x^3} \frac{1}{1 + \ln t} dt$  for  $x \geq 1$ , then  $f'(2) =$

(A)  $\frac{1}{1 + \ln 2}$

(B)  $\frac{12}{1 + \ln 2}$

(C)  $\frac{1}{1 + \ln 8}$

(D)  $\frac{12}{1 + \ln 8}$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.3A:</b> Analyze functions defined by an integral.	<b>EK 3.3A2:</b> If $f$ is a continuous function on the interval $[a, b]$ , then $\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$ , where $x$ is between $a$ and $b$ .	<b>MPAC 1:</b> Reasoning with definitions and theorems <b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C4:</b> The chain rule provides a way to differentiate composite functions.	

8. Which of the following limits is equal to  $\int_3^5 x^4 dx$ ?

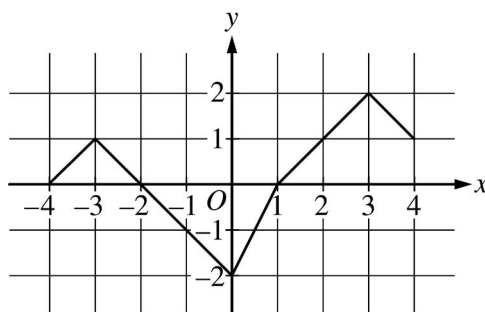
(A)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{k}{n}\right)^4 \frac{1}{n}$

(B)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{k}{n}\right)^4 \frac{2}{n}$

(C)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{2k}{n}\right)^4 \frac{1}{n}$

(D)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(3 + \frac{2k}{n}\right)^4 \frac{2}{n}$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.2A(a):</b> Interpret the definite integral as the limit of a Riemann sum.	<b>EK 3.2A3:</b> The information in a definite integral can be translated into the limit of a related Riemann sum, and the limit of a Riemann sum can be written as a definite integral.	<b>MPAC 1:</b> Reasoning with definitions and theorems  <b>MPAC 5:</b> Building notational fluency

Graph of  $f$ 

9. The function  $f$  is continuous for  $-4 \leq x \leq 4$ . The graph of  $f$  shown above consists of five line segments. What is the average value of  $f$  on the interval  $-4 \leq x \leq 4$ ?

- (A)  $\frac{1}{8}$   
 (B)  $\frac{3}{16}$   
 (C)  $\frac{15}{16}$   
 (D)  $\frac{3}{2}$

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.4B:</b> Apply definite integrals to problems involving the average value of a function.	<b>EK 3.4B1:</b> The average value of a function $f$ over an interval $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$ .	<b>MPAC 1:</b> Reasoning with definitions and theorems
<b>LO 3.2C:</b> Calculate a definite integral using areas and properties of definite integrals.	<b>EK 3.2C1:</b> In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.	<b>MPAC 4:</b> Connecting multiple representations

$t$	0	2
$f(t)$	4	12

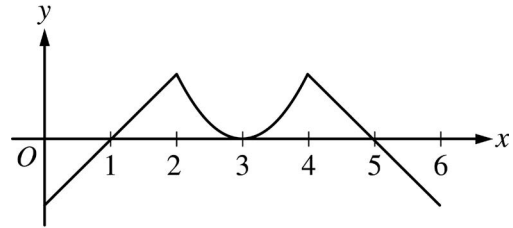
10. Let  $y = f(t)$  be a solution to the differential equation  $\frac{dy}{dt} = ky$ , where  $k$  is a constant. Values of  $f$  for selected values of  $t$  are given in the table above. Which of the following is an expression for  $f(t)$ ?

- (A)  $4e^{\frac{t}{2} \ln 3}$   
 (B)  $e^{\frac{t}{2} \ln 9} + 3$   
 (C)  $2t^2 + 4$   
 (D)  $4t + 4$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.5B:</b> Interpret, create and solve differential equations from problems in context.	<b>EK 3.5B1:</b> The model for exponential growth and decay that arises from the statement “The rate of change of a quantity is proportional to the size of the quantity” is $\frac{dy}{dt} = ky$ .	<b>MPAC 3:</b> Implementing algebraic/computational processes  <b>MPAC 4:</b> Connecting multiple representations

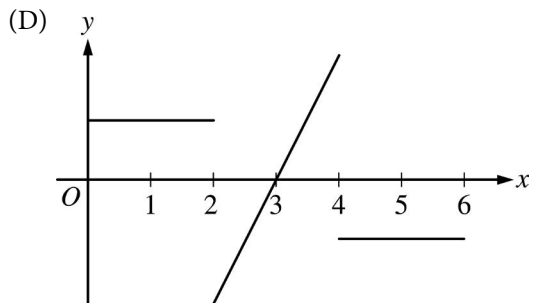
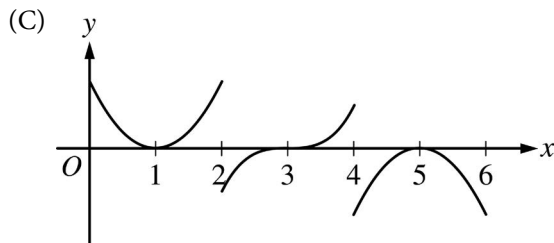
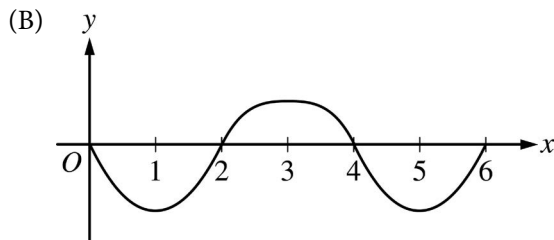
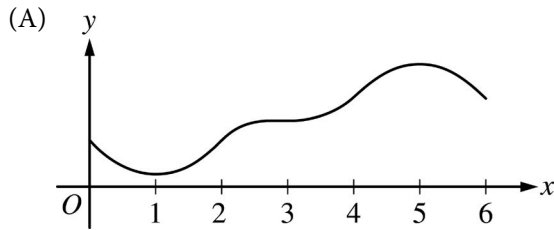
**Multiple Choice: Section I, Part B**

A graphing calculator is required for some questions on this part of the exam.



Graph of  $f'$

11. The graph of  $f'$ , the derivative of the function  $f$ , is shown above. Which of the following could be the graph of  $f$ ?





Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.2A:</b> Use derivatives to analyze properties of a function.	<b>EK 2.2A3:</b> Key features of the graphs of $f$ , $f'$ , and $f''$ are related to one another.	<b>MPAC 4:</b> Connecting multiple representations
<b>LO 2.2B:</b> Recognize the connection between differentiability and continuity.	<b>EK 2.2B2:</b> If a function is differentiable at a point, then it is continuous at that point.	<b>MPAC 2:</b> Connecting concepts

12. The derivative of a function  $f$  is given by  $f'(x) = e^{\sin x} - \cos x - 1$  for  $0 < x < 9$ . On what intervals is  $f$  decreasing?
- (A)  $0 < x < 0.633$  and  $4.115 < x < 6.916$
- (B)  $0 < x < 1.947$  and  $5.744 < x < 8.230$
- (C)  $0.633 < x < 4.115$  and  $6.916 < x < 9$
- (D)  $1.947 < x < 5.744$  and  $8.230 < x < 9$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.2A:</b> Use derivatives to analyze properties of a function.	<b>EK 2.2A1:</b> First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	<b>MPAC 4:</b> Connecting multiple representations <b>MPAC 2:</b> Connecting concepts

13. The temperature of a room, in degrees Fahrenheit, is modeled by  $H$ , a differentiable function of the number of minutes after the thermostat is adjusted. Of the following, which is the best interpretation of  $H'(5) = 2$ ?
- (A) The temperature of the room is 2 degrees Fahrenheit, 5 minutes after the thermostat is adjusted.
- (B) The temperature of the room increases by 2 degrees Fahrenheit during the first 5 minutes after the thermostat is adjusted.
- (C) The temperature of the room is increasing at a constant rate of  $\frac{2}{5}$  degree Fahrenheit per minute.
- (D) The temperature of the room is increasing at a rate of 2 degrees Fahrenheit per minute, 5 minutes after the thermostat is adjusted.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.3A:</b> Interpret the meaning of a derivative within a problem.	<b>EK 2.3A1:</b> The unit for $f'(x)$ is the unit for $f$ divided by the unit for $x$ .	<b>MPAC 2:</b> Connecting concepts
<b>LO 2.3D:</b> Solve problems involving rates of change in applied contexts.	<b>EK 2.3D1:</b> The derivative can be used to express information about rates of change in applied contexts.	<b>MPAC 5:</b> Building notational fluency

14. A function  $f$  is continuous on the closed interval  $[2, 5]$  with  $f(2) = 17$  and  $f(5) = 17$ . Which of the following additional conditions guarantees that there is a number  $c$  in the open interval  $(2, 5)$  such that  $f'(c) = 0$ ?
- (A) No additional conditions are necessary.
- (B)  $f$  has a relative extremum on the open interval  $(2, 5)$ .
- (C)  $f$  is differentiable on the open interval  $(2, 5)$ .
- (D)  $\int_2^5 f(x) dx$  exists.

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.4A:</b> Apply the Mean Value Theorem to describe the behavior of a function over an interval.	<b>EK 2.4A1:</b> If a function $f$ is continuous over the interval $[a, b]$ and differentiable over the interval $(a, b)$ , the Mean Value Theorem guarantees a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.	<b>MPAC 1:</b> Reasoning with definitions and theorems <b>MPAC 5:</b> Building notational fluency

15. A rain barrel collects water off the roof of a house during three hours of heavy rainfall. The height of the water in the barrel increases at the rate of  $r(t) = 4t^3 e^{-1.5t}$  feet per hour, where  $t$  is the time in hours since the rain began. At time  $t = 1$  hour, the height of the water is 0.75 foot. What is the height of the water in the barrel at time  $t = 2$  hours?
- (A) 1.361 ft  
 (B) 1.500 ft  
 (C) 1.672 ft  
 (D) 2.111 ft

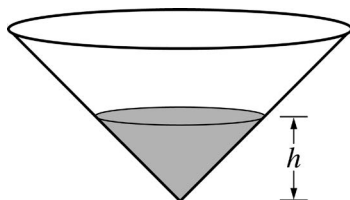
Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.4E:</b> Use the definite integral to solve problems in various contexts.	<b>EK 3.4E1:</b> The definite integral can be used to express information about accumulation and net change in many applied contexts.	<b>MPAC 2:</b> Connecting concepts
<b>LO 3.3B(b):</b> Evaluate definite integrals.	<b>EK 3.3B2:</b> If $f$ is continuous on the interval $[a, b]$ and $F$ is an antiderivative of $f$ , then $\int_a^b f(x) dx = F(b) - F(a)$ .	<b>MPAC 3:</b> Implementing algebraic/computational processes

16. A race car is traveling on a straight track at a velocity of 80 meters per second when the brakes are applied at time  $t = 0$  seconds. From time  $t = 0$  to the moment the race car stops, the acceleration of the race car is given by  $a(t) = -6t^2 - t$  meters per second per second. During this time period, how far does the race car travel?
- (A) 188.229 m  
 (B) 198.766 m  
 (C) 260.042 m  
 (D) 267.089 m

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.4C:</b> Apply definite integrals to problems involving motion.	<b>EK 3.4C1:</b> For a particle in rectilinear motion over an interval of time, the definite integral of velocity represents the particle's displacement over the interval of time, and the definite integral of speed represents the particle's total distance traveled over the interval of time.	<b>MPAC 2:</b> Connecting concepts  <b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 3.1A:</b> Recognize antiderivatives of basic functions.	<b>EK 3.1A2:</b> Differentiation rules provide the foundation for finding antiderivatives.	

**Free Response: Section II, Part A**

A graphing calculator is required for problems on this part of the exam.



1. The height of the water in a conical storage tank, shown above, is modeled by a differentiable function  $h$ , where  $h(t)$  is measured in meters and  $t$  is measured in hours. At time  $t = 0$ , the height of the water in the tank is 25 meters. The height is changing at the rate
- $$h'(t) = 2 - \frac{24e^{-0.025t}}{t + 4}$$
- meters per hour for  $0 \leq t \leq 24$ .
- (a) When the height of the water in the tank is  $h$  meters, the volume of water is  $V = \frac{1}{3}\pi h^3$ . At what rate is the volume of water changing at time  $t = 0$ ? Indicate units of measure.
- (b) What is the minimum height of the water during the time period  $0 \leq t \leq 24$ ? Justify your answer.
- (c) The line tangent to the graph of  $h$  at  $t = 16$  is used to approximate the height of the water in the tank. Using the tangent line approximation, at what time  $t$  does the height of the water return to 25 meters?

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C5:</b> The chain rule is the basis for implicit differentiation.	<b>MPAC 1:</b> Reasoning with definitions and theorems
<b>LO 2.3A:</b> Interpret the meaning of a derivative within a problem.	<b>EK 2.3A1:</b> The unit for $f'(x)$ is the unit for $f$ divided by the unit for $x$ .	<b>MPAC 2:</b> Connecting concepts
<b>LO 2.3B:</b> Solve problems involving the slope of a tangent line.	<b>EK 2.3B2:</b> The tangent line is the graph of a locally linear approximation of the function near the point of tangency.	<b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 2.3C:</b> Solve problems involving related rates, optimization, rectilinear motion, (BC) and planar motion.	<b>EK 2.3C2:</b> The derivative can be used to solve related rates problems, that is, finding a rate at which one quantity is changing by relating it to other quantities whose rates of change are known.	<b>MPAC 5:</b> Building notational fluency <b>MPAC 6:</b> Communicating

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.3C:</b> Solve problems involving related rates, optimization, rectilinear motion, <i>(BC)</i> and planar motion.	<b>EK 2.3C3:</b> The derivative can be used to solve optimization problems, that is, finding a maximum or minimum value of a function over a given interval.	
<b>LO 3.3B(b):</b> Evaluate definite integrals.	<b>EK 3.3B2:</b> If $f$ is continuous on the interval $[a, b]$ and $F$ is an antiderivative of $f$ , then $\int_a^b f(x) dx = F(b) - F(a)$ .	
<b>LO 3.4E:</b> Use the definite integral to solve problems in various contexts.	<b>EK 3.4E1:</b> The definite integral can be used to express information about accumulation and net change in many applied contexts.	

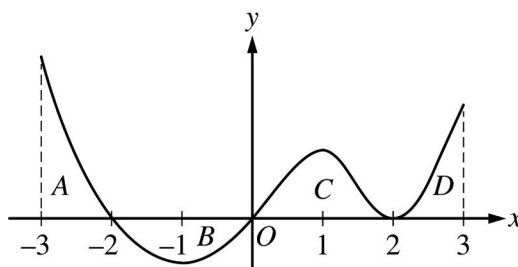
To answer Question 1 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- ▶ Engage in **reasoning with theorems** (MPAC 1) in order to find the derivative of volume with respect to time as well as in using the Fundamental Theorem of Calculus to find  $h(t)$  for particular values of  $t$ .
- ▶ **Connect the concept** (MPAC 2) of derivative to both the concept of optimization and the concept of slope of a tangent line.
- ▶ Use proper **notational fluency** (MPAC 5) to **communicate** (MPAC 6) the process of finding the values for  $h(24)$  and  $h(6.261)$  and to interpret the meaning of  $h'(t)$ .
- ▶ Use **algebraic manipulation** (MPAC 3) to substitute  $\frac{dh}{dt}$  into the expression for  $\frac{dV}{dt}$  and find the equation of a tangent line.



**Free Response: Section II, Part B**

No calculator is allowed for problems on this part of the exam.

Graph of  $f$ 

2. The graph of a differentiable function  $f$  is shown above for  $-3 \leq x \leq 3$ . The graph of  $f$  has horizontal tangent lines at  $x = -1$ ,  $x = 1$ , and  $x = 2$ . The areas of regions  $A$ ,  $B$ ,  $C$ , and  $D$  are 5, 4, 5, and 3, respectively. Let  $g$  be the antiderivative of  $f$  such that  $g(3) = 7$ .
- Find all values of  $x$  on the open interval  $-3 < x < 3$  for which the function  $g$  has a relative maximum. Justify your answer.
  - On what open intervals contained in  $-3 < x < 3$  is the graph of  $g$  concave up? Give a reason for your answer.
  - Find the value of  $\lim_{x \rightarrow 0} \frac{g(x)+1}{2x}$ , or state that it does not exist. Show the work that leads to your answer.
  - Let  $h$  be the function defined by  $h(x) = 3f(2x+1) + 4$ . Find the value of  $\int_{-2}^1 h(x) dx$ .

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 1.1C:</b> Determine limits of functions.	<b>EK 1.1C3:</b> Limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ may be evaluated using L'Hospital's Rule.	<b>MPAC 1:</b> Reasoning with definitions and theorems
<b>LO 1.2A:</b> Analyze functions for intervals of continuity or points of discontinuity.	<b>EK 1.2A1:</b> A function $f$ is continuous at $x = c$ provided that $f(c)$ exists, $\lim_{x \rightarrow c} f(x)$ exists, and $\lim_{x \rightarrow c} f(x) = f(c)$ .	<b>MPAC 2:</b> Connecting concepts
<b>LO 2.2A:</b> Use derivatives to analyze properties of a function.	<b>EK 2.2A1:</b> First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	<b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 2.2B:</b> Recognize the connection between differentiability and continuity.	<b>EK 2.2B2:</b> If a function is differentiable at a point, then it is continuous at that point.	<b>MPAC 4:</b> Connecting multiple representations
		<b>MPAC 5:</b> Building notational fluency
		<b>MPAC 6:</b> Communicating

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.2C:</b> Calculate a definite integral using areas and properties of definite integrals.	<b>EK 3.2C1:</b> In some cases, a definite integral can be evaluated by using geometry and the connection between the definite integral and area.	
<b>LO 3.2C:</b> Calculate a definite integral using areas and properties of definite integrals.	<b>EK 3.2C2:</b> Properties of definite integrals include the integral of a constant times a function, the integral of the sum of two functions, reversal of limits of integration, and the integral of a function over adjacent intervals.	
<b>LO 3.3B(b):</b> Evaluate definite integrals.	<b>EK 3.3B5:</b> Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, (BC) integration by parts, and nonrepeating linear partial fractions.	

To answer Question 2 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- ▶ **Reason with definitions and theorems** (MPAC 1) by applying the Fundamental Theorem of Calculus and the concept of area to find the integral over specific intervals.
- ▶ Confirm that the hypotheses have been satisfied when applying L'Hospital's rule to find a limit. Correctly using L'Hospital's rule involves **manipulating algebraic** (MPAC 3) quantities.
- ▶ **Connect the concepts** (MPAC 2) of a function and its derivative to identify a maximum value and to determine concavity, and **connect the concepts** (MPAC 2) of continuity and limit to find  $g(0)$ .
- ▶ **Connect the graphical representation** (MPAC 4) of a function to the words describing certain attributes of the function and to a symbolic description involving the function.
- ▶ Extract information from the graph of  $f(x)$  to **compute** (MPAC 3) definite integrals for  $f$  and  $h$  over specified intervals.
- ▶ Build **notational fluency** (MPAC 5) when using integration by substitution to find the integral of  $h(x) = 3f(2x + 1) + 4$  over an interval, including adjusting the endpoints of the interval.
- ▶ Clearly **communicate** (MPAC 6) the justification for why a critical point is a relative maximum and indicate the direction of concavity.

## Answers and Rubrics (AB)

### Answers to Multiple-Choice Questions

1.	B
2.	B
3.	C
4.	A
5.	A
6.	B
7.	D
8.	D
9.	B
10.	A
11.	A
12.	A
13.	D
14.	C
15.	D
16.	B

## Rubrics for Free-Response Questions

### Question 1

Solutions	Point Allocation								
<p>(a) <math>\frac{dV}{dt} = \frac{1}{3}\pi 3h^2 \frac{dh}{dt} = \pi h^2 \frac{dh}{dt}</math></p> <p>At <math>t = 0</math>,</p> $\frac{dV}{dt} = \pi(25)^2(-4) = -2500\pi = -7853.982 \text{ (or } -7853.981) \text{ cubic meters per hour.}$	$2 : \left\{ \begin{array}{l} 1 : \frac{dV}{dt} \\ 1 : \text{answer with units} \end{array} \right.$								
<p>(b) The absolute minimum must be at a critical point or an endpoint.</p> <p><math>h'(t) = 0</math> when <math>t = 6.261</math>.</p> $h(t) = 25 + \int_0^t h'(x) dx$ <table border="1" style="margin-left: auto; margin-right: auto;"> <thead> <tr> <th style="text-align: center;"><math>t</math></th> <th style="text-align: center;"><math>h(t)</math></th> </tr> </thead> <tbody> <tr> <td style="text-align: center;">0</td> <td style="text-align: center;">25</td> </tr> <tr> <td style="text-align: center;">6.261</td> <td style="text-align: center;">16.33873</td> </tr> <tr> <td style="text-align: center;">24</td> <td style="text-align: center;">34.56246</td> </tr> </tbody> </table> <p>The minimum height is 16.339 (or 16.338) meters.</p>	$t$	$h(t)$	0	25	6.261	16.33873	24	34.56246	$4 : \left\{ \begin{array}{l} 1 : \text{considers } h'(t) = 0 \\ 1 : \text{Fundamental Theorem} \\ \text{of Calculus} \\ 1 : \text{absolute minimum value} \\ 1 : \text{justification} \end{array} \right.$
$t$	$h(t)$								
0	25								
6.261	16.33873								
24	34.56246								
<p>(c) <math>h(16) = 25 + \int_0^{16} h'(t) dt = 23.49607</math></p> $h'(16) = 1.19562$ <p>An equation for the tangent line is</p> $y = 1.196(t - 16) + 23.496.$ <p><math>y = 25</math> when <math>t = 17.258</math> (or 17.257).</p>	$3 : \left\{ \begin{array}{l} 1 : h(16) \\ 1 : \text{tangent line equation} \\ 1 : \text{answer} \end{array} \right.$								

Question 2

Solutions	Point Allocation
(a) $g$ has a relative maximum at $x = -2$ since $g' = f$ changes sign from positive to negative at $x = -2$ .	1 : answer with justification
(b) The graph of $g$ is concave up for $-1 < x < 1$ and $2 < x < 3$ because $g' = f$ is increasing on those intervals.	2 : $\left\{ \begin{array}{l} 1 : \text{answer} \\ 1 : \text{reason} \end{array} \right.$
(c) Because $g$ is continuous at $x = 0$ , $\lim_{x \rightarrow 0} g(x) = g(0)$ . $g(3) = g(0) + \int_0^3 f(x) dx$ $g(0) = g(3) - \int_0^3 f(x) dx = 7 - (5 + 3) = -1$ $\lim_{x \rightarrow 0} g(x) + 1 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} 2x = 0.$ Using L'Hospital's Rule, $\lim_{x \rightarrow 0} \frac{g(x) + 1}{2x} = \lim_{x \rightarrow 0} \frac{g'(x)}{2} = \lim_{x \rightarrow 0} \frac{f(x)}{2} = \frac{f(0)}{2} = 0$	3 : $\left\{ \begin{array}{l} 1 : g(0) \\ 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{array} \right.$
(d) $\int_{-2}^1 h(x) dx = \int_{-2}^1 (3f(2x+1) + 4) dx = 3 \int_{-2}^1 f(2x+1) dx + \int_{-2}^1 4 dx$ Let $u = 2x + 1$ . Then $du = 2dx$ and $3 \int_{-2}^1 f(2x+1) dx + \int_{-2}^1 4 dx = \frac{3}{2} \int_{-3}^3 f(u) du + 12$ $= \frac{3}{2}(5 - 4 + 5 + 3) + 12 = 25.5$	3 : $\left\{ \begin{array}{l} 2 : \text{Fundamental Theorem} \\ \quad \text{of Calculus} \\ 1 : \text{answer} \end{array} \right.$

## AP Calculus BC Sample Exam Questions

### Multiple Choice: Section I, Part A

A calculator may not be used on questions on this part of the exam.

1. The position of a particle moving in the  $xy$ -plane is given by the parametric equations  $x(t) = \frac{6t}{t+1}$  and  $y(t) = \frac{-8}{t^2+4}$ . What is the slope of the line tangent to the path of the particle at the point where  $t = 2$ ?
- (A)  $\frac{1}{2}$
- (B)  $\frac{2}{3}$
- (C)  $\frac{3}{4}$
- (D)  $\frac{4}{3}$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C7:</b> (BC) Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.	<b>MPAC 3:</b> Implementing algebraic/computational processes <b>MPAC 2:</b> Connecting concepts

2. Let  $y = f(x)$  be the solution to the differential equation  $\frac{dy}{dx} = 1 + 2y$  with the initial condition  $f(0) = 1$ . What is the approximation for  $f(1)$  if Euler's method is used, starting at  $x = 0$  with a step size of 0.5?
- (A) 2.5  
 (B) 3.5  
 (C) 4.0  
 (D) 5.5

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.3F:</b> Estimate solutions to differential equations.	<b>EK 2.3F2:</b> (BC) For differential equations, Euler's method provides a procedure for approximating a solution or a point on a solution curve.	<b>MPAC 3:</b> Implementing algebraic/computational processes  <b>MPAC 2:</b> Connecting concepts

3. For what value of  $k$ , if any, is  $\int_0^{\infty} kxe^{-2x} dx = 1$ ?
- (A)  $\frac{1}{4}$
- (B) 1
- (C) 4
- (D) There is no such value of  $k$ .

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.2D:</b> (BC) Evaluate an improper integral or show that an improper integral diverges.	<b>EK 3.2D2:</b> (BC) Improper integrals can be determined using limits of definite integrals.	<b>MPAC 3:</b> Implementing algebraic/computational processes
<b>LO 3.3B(b):</b> Evaluate definite integrals.	<b>EK 3.3B5:</b> Techniques for finding antiderivatives include algebraic manipulation such as long division and completing the square, substitution of variables, (BC) integration by parts, and nonrepeating linear partial fractions.	<b>MPAC 1:</b> Reasoning with definitions and theorems



4. The Taylor series for a function  $f$  about  $x = 0$  converges to  $f$  for  $-1 \leq x \leq 1$ . The  $n$ th-degree Taylor polynomial for  $f$  about  $x = 0$  is given by  $P_n(x) = \sum_{k=1}^n (-1)^k \frac{x^k}{k^2 + k + 1}$ . Of the following,

which is the smallest number  $M$  for which the alternating series error bound guarantees that

$$|f(1) - P_4(1)| \leq M?$$

(A)  $\frac{1}{5!} \cdot \frac{1}{31}$

(B)  $\frac{1}{4!} \cdot \frac{1}{21}$

(C)  $\frac{1}{31}$

(D)  $\frac{1}{21}$

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**Learning Objectives**

**LO 4.1B:** Determine or estimate the sum of a series.

**LO 4.2B:** Write a power series representing a given function.

**Essential Knowledge**

**EK 4.1B2:** If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.

**EK 4.2B4:** A Taylor polynomial for  $f(x)$  is a partial sum of the Taylor series for  $f(x)$ .

**Mathematical Practices for AP Calculus**

**MPAC 1:** Reasoning with definitions and theorems

**MPAC 5:** Building notational fluency

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$x$	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-2	1	4
1	2	-3	3	-2
2	-1	1	4	5

5. Selected values of a function  $f$  and its first three derivatives are indicated in the table above. What is the third-degree Taylor polynomial for  $f$  about  $x = 1$ ?

- (A)  $2 - 3x + \frac{3}{2}x^2 - \frac{1}{3}x^3$
- (B)  $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$
- (C)  $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{2}{3}(x-1)^3$
- (D)  $2 - 3(x-1) + 3(x-1)^2 - 2(x-1)^3$

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 4.2A:</b> Construct and use Taylor polynomials.	<b>EK 4.2A1:</b> The coefficient of the $n$ th-degree term in a Taylor polynomial centered at $x = a$ for the function $f$ is $\frac{f^{(n)}(a)}{n!}$ .	<b>MPAC 1:</b> Reasoning with definitions and theorems  <b>MPAC 4:</b> Connecting multiple representations

6. Which of the following statements about the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$  is true?
- (A) The series converges absolutely.
- (B) The series converges conditionally.
- (C) The series converges but neither conditionally nor absolutely.
- (D) The series diverges.

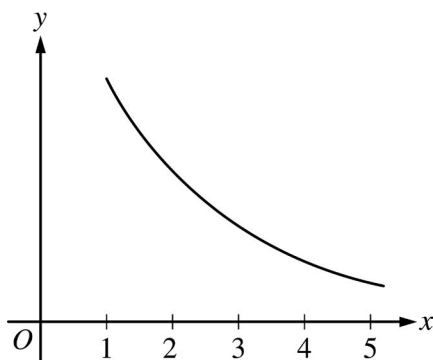
Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 4.1A:</b> Determine whether a series converges or diverges.	<b>EK 4.1A4:</b> A series may be absolutely convergent, conditionally convergent, or divergent.	<b>MPAC 1:</b> Reasoning with definitions and theorems  <b>MPAC 5:</b> Building notational fluency

**Multiple Choice: Section I, Part B**

A graphing calculator is required for some questions on this part of the exam.

7. At time  $t \geq 0$ , a particle moving in the  $xy$ -plane has velocity vector given by  $v(t) = \langle 4e^{-t}, \sin(1 + \sqrt{t}) \rangle$ . What is the total distance the particle travels between  $t = 1$  and  $t = 3$ ?
- (A) 1.861  
 (B) 1.983  
 (C) 2.236  
 (D) 4.851

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 3.4C:</b> Apply definite integrals to problems involving motion.	<b>EK 3.4C2:</b> (BC)The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.	<b>MPAC 2:</b> Connecting concepts <b>MPAC 3:</b> Implementing algebraic/computational processes

Graph of  $g$ 

8. For  $x \geq 1$ , the continuous function  $g$  is decreasing and positive. A portion of the graph of  $g$  is shown above. For  $n \geq 1$ , the  $n$ th term of the series  $\sum_{n=1}^{\infty} a_n$  is defined by  $a_n = g(n)$ . If

$\int_1^{\infty} g(x) dx$  converges to 8, which of the following could be true?

- (A)  $\sum_{n=1}^{\infty} a_n = 6$
- (B)  $\sum_{n=1}^{\infty} a_n = 8$
- (C)  $\sum_{n=1}^{\infty} a_n = 10$
- (D)  $\sum_{n=1}^{\infty} a_n$  diverges

Learning Objective	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 4.1A:</b> Determine whether a series converges or diverges.	<b>EK 4.1A6:</b> In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the $n$ th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	<b>MPAC 1:</b> Reasoning with definitions and theorems  <b>MPAC 4:</b> Connecting multiple representations

**Free Response: Section II, Part A**

A graphing calculator is required for problems on this part of the exam.

1. At time  $t \geq 0$ , the position of a particle moving along a curve in the  $xy$ -plane is  $(x(t), y(t))$ , where  $\frac{dx}{dt} = t - 5 \cos t$  and  $\frac{dy}{dt} = 6 \cos(1 + \sin t)$ . At time  $t = 3$ , the particle is at position  $(-1, 2)$ .
- Write an equation for the line tangent to the path of the particle at time  $t = 3$ .
  - Find the time  $t$  when the line tangent to the path of the particle is vertical. Is the direction of motion of the particle up or down at that moment? Give a reason for your answer.
  - Find the  $y$ -coordinate of the particle's position at time  $t = 0$ .
  - Find the total distance traveled by the particle for  $0 \leq t \leq 3$ .

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 2.1C:</b> Calculate derivatives.	<b>EK 2.1C7:</b> (BC) Methods for calculating derivatives of real-valued functions can be extended to vector-valued functions, parametric functions, and functions in polar coordinates.	<b>MPAC 1:</b> Reasoning with definitions and theorems <b>MPAC 2:</b> Connecting concepts
<b>LO 2.2A:</b> Use derivatives to analyze properties of a function.	<b>EK 2.2A1:</b> First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	<b>MPAC 3:</b> Implementing algebraic/computational processes <b>MPAC 5:</b> Building notational fluency
<b>LO 2.3B:</b> Solve problems involving the slope of a tangent line.	<b>EK 2.3B1:</b> The derivative at a point is the slope of the line tangent to a graph at that point on the graph.	<b>MPAC 6:</b> Communicating
<b>LO 2.3C:</b> Solve problems involving related rates, optimization, rectilinear motion, (BC) and planar motion.	<b>EK 2.3C4:</b> (BC) Derivatives can be used to determine velocity, speed, and acceleration for a particle moving along curves given by parametric or vector-valued functions.	
<b>LO 3.3B(b):</b> Evaluate definite integrals.	<b>EK 3.3B2:</b> If $f$ is continuous on the interval $[a, b]$ and $F$ is an antiderivative of $f$ , then $\int_a^b f(x) dx = F(b) - F(a)$ .	
<b>LO 3.4C:</b> Apply definite integrals to problems involving motion.	<b>EK 3.4C2:</b> (BC) The definite integral can be used to determine displacement, distance, and position of a particle moving along a curve given by parametric or vector-valued functions.	

To answer Question 1 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- ▶ Engage in **reasoning with definitions and theorems** (MPAC 1) when finding the total distance traveled.
- ▶ **Connect the concepts** (MPAC 2) of derivative and position of a particle as well as the concepts of vertical tangent lines and motion.
- ▶ Use **algebraic manipulation** (MPAC 3) to find  $\frac{dy}{dx}$  and the equation of a tangent line.
- ▶ Build **notational fluency** (MPAC 5) by expressing  $\frac{dy}{dx}$  in terms of  $\frac{dy}{dt}$  and  $\frac{dx}{dt}$  and in **communicating** (MPAC 6) the process that leads to finding the  $y$ -coordinate and the total distance.
- ▶ **Communicate** (MPAC 6) using accurate and precise language and **notation** (MPAC 5) in reporting information provided by technology and in explaining what the sign of  $\left. \frac{dy}{dt} \right|_{t=3}$  implies about the vertical direction of motion of the particle.

**Free Response: Section II, Part B**

No calculator is allowed for problems on this part of the exam.

2. The function  $f$  has derivatives of all orders at  $x = 0$ , and the Maclaurin series for  $f$  is

$$\sum_{n=2}^{\infty} \frac{\ln n}{3^n n^3} x^n.$$

- (a) Find  $f'(0)$  and  $f^{(4)}(0)$ .
- (b) Does  $f$  have a relative minimum, a relative maximum, or neither at  $x = 0$ ? Justify your answer.
- (c) Using the ratio test, determine the interval of convergence of the Maclaurin series for  $f$ . Justify your answer.

Learning Objectives	Essential Knowledge	Mathematical Practices for AP Calculus
<b>LO 1.1A(b):</b> Interpret limits expressed symbolically.	<b>EK 1.1A2:</b> The concept of a limit can be extended to include one-sided limits, limits at infinity, and infinite limits.	<b>MPAC 1:</b> Reasoning with definitions and theorems <b>MPAC 2:</b> Connecting concepts
<b>LO 2.2A:</b> Use derivatives to analyze properties of a function.	<b>EK 2.2A1:</b> First and second derivatives of a function can provide information about the function and its graph including intervals of increase or decrease, local (relative) and global (absolute) extrema, intervals of upward or downward concavity, and points of inflection.	<b>MPAC 3:</b> Implementing algebraic/computational processes <b>MPAC 5:</b> Building notational fluency
<b>LO 4.1A:</b> Determine whether a series converges or diverges.	<b>EK 4.1A3:</b> Common series of numbers include geometric series, the harmonic series, and $p$ -series.	<b>MPAC 6:</b> Communicating
<b>LO 4.1A:</b> Determine whether a series converges or diverges.	<b>EK 4.1A5:</b> If a series converges absolutely, then it converges.	
<b>LO 4.1A:</b> Determine whether a series converges or diverges.	<b>EK 4.1A6:</b> In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the $n$ th term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.	
<b>LO 4.2A:</b> Construct and use Taylor polynomials.	<b>EK 4.2A1:</b> The coefficient of the $n$ th-degree term in a Taylor polynomial centered at $x = a$ for the function $f$ is $\frac{f^{(n)}(a)}{n!}$ .	
<b>LO 4.2C:</b> Determine the radius and interval of convergence of a power series.	<b>EK 4.2C2:</b> The ratio test can be used to determine the radius of convergence of a power series.	



To answer Question 2 successfully, students must apply the Mathematical Practices for AP Calculus as described below:

- ▶ Engage in **reasoning with the definition** (MPAC 1) of the coefficients of the Maclaurin series to find the coefficients for  $f'(0)$  and  $f^{(4)}(0)$  and in applying the ratio test and comparison test to determine convergence.
- ▶ **Connect the concepts** (MPAC 2) of the first and second derivative to find a relative minimum and the concepts of convergence and absolute convergence when finding the interval of convergence.
- ▶ Use **algebraic manipulation** (MPAC 3) including working with logarithms and functions to find specific coefficients in the Maclaurin series, the limit in the ratio test, and the interval of convergence.
- ▶ Display facility with **notation** (MPAC 5) in **communicating** (MPAC 6) the justification for why  $f$  has a relative minimum and what constitutes the interval of convergence.

## Answers and Rubrics (BC)

### Answers to Multiple-Choice Questions

1.	C
2.	D
3.	C
4.	C
5.	B
6.	B
7.	A
8.	C

## Rubrics for Free-Response Questions

### Question 1

Solutions	Point Allocation
<p>(a) <math>\left. \frac{dy}{dx} \right _{t=3} = \left. \frac{dy/dt}{dx/dt} \right _{t=3} = \left. \frac{6 \cos(1 + \sin t)}{t - 5 \cos t} \right _{t=3} = 0.314</math></p> <p>An equation for the tangent line is <math>y = 2 + 0.314(x + 1)</math>.</p>	$2 : \begin{cases} 1 : \text{considers } \frac{dy}{dx} \text{ at } t = 3 \\ 1 : \text{tangent line equation} \end{cases}$
<p>(b) The tangent line is vertical when <math>\frac{dx}{dt} = 0</math> and <math>\frac{dy}{dt} \neq 0</math>.</p> <p><math>\frac{dx}{dt} = 0</math> when <math>t = 1.30644</math>.</p> <p>Because <math>y'(1.30644) = -2.305884 &lt; 0</math>, the <math>y</math>-coordinate is decreasing and so the particle is moving down at that moment.</p>	$3 : \begin{cases} 1 : \text{considers } \frac{dx}{dt} = 0 \\ 1 : t = 1.30644 \\ 1 : \text{conclusion with reason} \end{cases}$
<p>(c) <math>y(3) = y(0) + \int_0^3 y'(t) dt</math></p> <p><math>y(0) = y(3) - \int_0^3 y'(t) dt = y(3) + 1.63359 = 3.634</math> (or 3.633)</p>	$2 : \begin{cases} 1 : \text{Fundamental Theorem} \\ \quad \text{of Calculus} \\ 1 : \text{answer} \end{cases}$
<p>(d) Distance <math>= \int_0^3 \sqrt{(x'(t))^2 + (y'(t))^2} dt = 13.453</math></p>	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

Solutions	Point Allocation
<p>(a) <math>\frac{f'(0)}{1!} = a_1 = 0 \Rightarrow f'(0) = 0</math></p> $\frac{f^{(4)}(0)}{4!} = a_4 = \frac{\ln 4}{3^4 4^3} \Rightarrow f^{(4)}(0) = \frac{\ln 4}{3^4 4^3} \cdot 4! = \frac{\ln 4}{216}$	$2 : \begin{cases} 1 : f'(0) \\ 1 : f^{(4)}(0) \end{cases}$
<p>(b) <math>f'(0) = 0</math></p> $\frac{f''(0)}{2!} = a_2 = \frac{\ln 2}{3^2 2^3} \Rightarrow f''(0) = \frac{\ln 2}{3^2 2^3} \cdot 2! = \frac{\ln 2}{36} > 0$ <p>By the Second Derivative Test, <math>f</math> has a relative minimum at <math>x = 0</math>.</p>	$2 : \begin{cases} 1 : \text{considers } f''(0) \\ 1 : \text{answer with justification} \end{cases}$
<p>(c) Using the ratio test,</p> $\lim_{n \rightarrow \infty} \left  \frac{\frac{\ln(n+1)}{3^{n+1} (n+1)^3} x^{n+1}}{\frac{\ln n}{3^n n^3} x^n} \right  = \lim_{n \rightarrow \infty} \left  \frac{\ln(n+1)}{\ln n} \cdot \left(\frac{n}{n+1}\right)^3 \cdot \frac{x}{3} \right  = \left  \frac{x}{3} \right  < 1$ <p><math> x  &lt; 3</math>, therefore the radius of convergence is <math>R = 3</math>, and the series converges on the interval <math>-3 &lt; x &lt; 3</math>.</p> <p>When <math>x = 3</math>, the series is <math>\sum_{n=2}^{\infty} \frac{\ln n}{n^3}</math>.</p> <p>Because <math>0 &lt; \frac{\ln n}{n^3} &lt; \frac{n}{n^3} = \frac{1}{n^2}</math> for all <math>n \geq 2</math> and the <math>p</math>-series <math>\sum_{n=2}^{\infty} \frac{1}{n^2}</math> converges, the series <math>\sum_{n=2}^{\infty} \frac{\ln n}{n^3}</math> converges by the comparison test.</p>	$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

When  $x = -3$ , the series is  $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n^3}$ .

This series is absolutely convergent because  $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$  converges.

The interval of convergence is  $-3 \leq x \leq 3$ .

